Chapter 4

Baire's Theorem and Applications

4.1 Baire's Theorem

Baire's theorem is a result on complete metric spaces which will be used in this chapter to prove some very important results on Banach spaces.

Theorem 4.1.1 (**Baire's Theorem**) *Let (X, d) be a complete metric space. Let $\{V_n\}_{n=1}^{\infty}$ be a collection of open dense sets. Then*

$$\cap_{n=1}^{\infty} V_n$$

is also dense.

Proof: Let W be any non-empty open set in X. We need to show that the intersection $\cap_n V_n$ has a point in W.

Since V_1 is dense, it follows that $W \cap V_1 \neq \emptyset$. Thus, we can find a point x_1 in this intersection (which is also an open set). Hence, there exists $r_1 > 0$ such that the open ball $B(x_1; r_1)$, with centre at x_1 and radius r_1, is contained in $W \cap V_1$. By shrinking r if necessary, we may also assume that the closure of this ball $\overline{B(x_1; r_1)}$ is also a subset of $W \cap V_1$ and that $0 < r_1 < 1$.

If $n \geq 2$, assume that we have chosen x_{n-1} and r_{n-1} suitably. The denseness of V_n shows that $V_n \cap B(x_{n-1}; r_{n-1}) \neq \emptyset$ and so we can choose x_n and r_n such that $0 < r_n < 1/n$ and

$$\overline{B(x_n; r_n)} \subset V_n \cap B(x_{n-1}; r_{n-1}).$$

Thus we now have a sequence of points $\{x_n\}$ in X. If $i > n$ and $j > n$, it is clear that both x_i and x_j both lie in $B(x_n; r_n)$ and so $d(x_i, x_j) < 2r_n < 2/n$ and thus, the sequence $\{x_n\}$ is Cauchy. Since X is complete, it follows that there exists $x \in X$ such that $x_n \to x$.

Now, for $i > n$, it follows that $x_i \in B(x_n; r_n)$ and so $x \in \overline{B(x_n; r_n)}$ for all n and so $x \in V_n$ for all n. Also $x \in \overline{B(x_1; r_1)}$ and so $x \in W$ as well. This completes the proof. ■

Remark 4.1.1 The main significance of this theorem is that, in particular, the intersection of a countable collection of non-empty open dense sets in a (non-empty) complete metric space is, again, non-empty. ■

Remark 4.1.2 An equivalent way of stating this theorem is that a complete metric space cannot be the countable union of nowhere dense sets. In the literature, countable unions of nowhere dense sets are said to be of the *first category* and all other sets are said to be of the *second category*. Thus, Baire's theorem states that every complete metric space is of second category and so, often in the literature, it is referred to as the *Baire category theorem.* ■

A countable intersection of open sets in a topological space is called a G_δ set. Since the countable union of countable sets is again countable, the following corallary is an immediate consequence of Baire's theorem.

Corollary 4.1.1 *In a complete metric space, the intersection of any countable collection of dense G_δ sets is again a dense G_δ set.* ■

Corollary 4.1.2 *In a complete metric space which has no isolated points, a countable dense set can never be a G_δ set.*

Proof: Let $E = \{x_k\}_{k=1}^\infty$ be a countable dense set in a complete metric space X. If it is a G_δ set, then, there exist open sets V_n such that

$$E = \cap_{n=1}^\infty V_n.$$

Clearly, since $E \subset V_n$ for each n, each set V_n is dense as well. Set

$$W_n = V_n \backslash \cup_{k=1}^n \{x_k\}.$$

Then each W_n is also dense and open, since X has no isolated points. But $\cap_{n=1}^\infty W_n = \emptyset$, which contradicts Baire's theorem. Hence the result. ■

4.2 Principle of Uniform Boundedness

As a first application of Baire's theorem we prove the following result.

Theorem 4.2.1 (**Banach-Steinhaus Theorem**) *Let V be a Banach space and let W be a normed linear space. Let I be an arbitrary indexing set and , for each $i \in I$, let $T_i \in \mathcal{L}(V, W)$. Then, either there exists $M > 0$ such that*

$$\|T_i\| \ \leq \ M, \quad \textit{for all} \ \ i \in I$$

or,

$$\sup_{i \in I} \|T_i(x)\| \ = \ \infty$$

for all x belonging to some dense G_δ set in V.

Proof: For each $x \in V$, set

$$\varphi(x) \ = \ \sup_{i \in I} \|T_i(x)\|.$$

Let

$$V_n \ = \ \{x \in V \mid \varphi(x) > n\}.$$

Since each T_i is continuous and since the norm is a continuous function, it is easy to see that V_n is open for each n.

Assume now that there exists N such that V_N fails to be dense in V. Then, there exists $x_0 \in V$ and $r > 0$ such that $x + x_0 \notin V_N$ if $\|x\| < r$. (In other words, there is an open ball $B(x_0; r)$, centered at x_0 and of radius r, which does not intersect V_N.) This implies that $\varphi(x + x_0) \leq N$ for all such x and so, for all $i \in I$,

$$\|T_i(x + x_0)\| \ \leq \ N.$$

Thus, if $\|x\| \leq r/2$, we have, for all $i \in I$,

$$\|T_i(x)\| \ \leq \ \|T_i(x + x_0)\| + \|T_i(x_0)\| \ \leq \ 2N.$$

It follows from this that, for all $i \in I$,

$$\|T_i\| \ \leq \ \frac{4N}{r}$$

and so the first alternative holds with $M = 4N/r$.

The other possibility is that each V_n is dense, and so, V being complete, by Baire's theorem, $\cap_n V_n$ is a dense G_δ and for each $x \in \cap_n V_n$, we have that $\varphi(x) = \infty$. This completes the proof. ■

An immediate consequence of the above result is the following.

Corollary 4.2.1 *If V is a Banach space and W is a normed linear space and if $T_i \in \mathcal{L}(V, W)$ for an indexing set I such that*

$$\sup_{i \in I} \|T_i(x)\| < \infty$$

for every $x \in V$, then there exists $M > 0$ such that

$$\|T_i\| \leq M \text{ for each } i \in I. \blacksquare$$

In other words, if the T_i are all pointwise bounded, then they are uniformly bounded in norm. For this reason, the Banach-Steinhaus theorem is also referred to as the *principle of uniform boundedness.*

Corollary 4.2.2 *Let V be a Banach space and let W be a normed linear space and let $\{T_n\}$ be a sequence of continuous linear transformations from V into W such that , for each $x \in V$, the sequence $\{T_n x\}$ is convergent in W. Define*

$$T(x) = \lim_{n \to \infty} T_n(x).$$

Then $T \in \mathcal{L}(V, W)$ and

$$\|T\| \leq \liminf_{n \to \infty} \|T_n\|. \tag{4.2.1}$$

Proof: It is clear that T is linear. By the Banach-Steinhaus theorem, it follows that $\{\|T_n\|\}$ is a bounded sequence. Let $\|T_n\| \leq C$ for all n. Then, for each $x \in V$ and for all n, we have

$$\|T_n(x)\| \leq C\|x\|.$$

Passing to the limit as $n \to \infty$, we deduce that

$$\|T(x)\| \leq C\|x\|$$

for each $x \in V$ and so $T \in \mathcal{L}(V, W)$. The relation (4.2.1) follows from the inequality

$$\|T_n(x)\| \leq \|T_n\| \, \|x\|$$

for each $x \in V$. $\blacksquare$

Corollary 4.2.3 *Let V be a Banach space and let $B \subset V$ be a subset. Assume that*

$$f(B) = \{f(x) \mid x \in B\}$$

is a bounded subset of the scalar field for each $f \in V^*$. Then B is a bounded subset of V.

Proof: For $x \in B$, consider the functional $J_x \in V^{**}$ defined by $J_x(f) = f(x)$ for $f \in V^*$. Then, we know that (cf. Corollary 3.1.2)

$$\|J_x\|_{V^{**}} = \|x\|_V.$$

Taking B as the indexing set and V^* as the Banach space, it follows from the Banach-Steinhaus theorem that $\|J_x\|$ is uniformly bounded in V^{**}, which is the same as saying that B is bounded in V. ■

Remark 4.2.1 To check the boundedness of a set V in a Banach space, it thus suffices to verify that its image under each continuous linear functional is bounded. In finite dimensional spaces, this is what we precisely do. We check that the image under each coordinate projection is bounded and these form a basis for the dual space. In the language of weak topologies (to be studied later), the conclusion of the preceding corollary is read as 'weakly bounded implies bounded'. ■

4.3 Application to Fourier Series

Let $f : [-\pi, \pi] \to \mathbb{R}$ be an integrable function. We can write its formal *Fourier series* (in exponential form) as follows:

$$f(t) \sim \sum_{n=-\infty}^{\infty} \widehat{f}(n) \exp(int)$$

where

$$\widehat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(s) \exp(-ins)\, ds \qquad (4.3.1)$$

are the *Fourier coefficients* of f. The first question that springs to the mind is "in what sense does the Fourier series of a function represent the function?" In particular, does the Fourier series of a continuous 2π-periodic function f converge to $f(t)$ at every point $t \in [-\pi, \pi]$? This is relevant since each term in the Fourier series is a continuous 2π-periodic function. Unfortunately, the answer is 'No!'. It was Dirichlet who first established (around 1829, nearly seven decades after a lengthy controversy began in Europe - about the validity of representing a function in terms of sines and cosines - and raged through the latter half of the eighteenth century) the sufficient conditions for the Fourier series of a function to converge to its value at a point. This was later strengthened by Jordan. In fact the study of the validity of Fourier expansions led to a lot of mathematical development such as making precise the notion of a function, Cantor's theory of infinite sets, the theories of integration by Riemann and by Lebesgue and theories of summability of series.

In this section, we will use the Banach-Steinhaus theorem to show that there exists a very large class of continuous 2π-periodic functions whose Fourier series fail to converge on a very large set of points.

To study the convergence of Fourier series, we need to study its partial sums:

$$s_m(f)(t) \;=\; \sum_{n=-m}^{m} \widehat{f}(n)\exp(int).$$

Using the formula for the Fourier coefficients (4.3.1), this becomes

$$s_m(f)(t) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} f(s)D_m(t-s)\,ds \qquad (4.3.2)$$

where

$$D_m(t) \;=\; \sum_{n=-m}^{m} \exp(int).$$

The function D_m is called the *Dirichlet kernel.* If we multiply $D_m(t)$ successively by $\exp(it/2)$ and $\exp(-it/2)$ and subtract, we see immediately that

$$D_m(t) \;=\; \begin{cases} \frac{\sin\left(m+\frac{1}{2}\right)t}{\sin\frac{t}{2}}, & \text{if } t \neq 2k\pi, \text{ for some non-negative integer } k \\ 2m+1, & \text{if } t = 2k\pi \text{ for some non-negative integer } k. \end{cases}$$

Proposition 4.3.1 *We have*

$$\lim_{n\to\infty}\int_{-\pi}^{\pi} |D_n(t)|\,dt \;=\; +\infty. \qquad (4.3.3)$$

Proof: For $t \in \mathbb{R}$, we have $|\sin t| \leq |t|$ and so

$$\begin{aligned}\int_{-\pi}^{\pi} |D_n(t)|\,dt \;&\geq\; 4\int_0^{\pi} \frac{|\sin(n+\frac{1}{2})t|}{t}\,dt \\ &=\; 4\int_0^{(n+\frac{1}{2})\pi} \frac{|\sin t|}{t}\,dt \\ &>\; 4\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi} \frac{|\sin t|}{t}\,dt\end{aligned}$$

$$> \quad 4\sum_{k=1}^{n}\int_{(k-1)\pi}^{k\pi}\frac{|\sin t|}{k\pi}\,dt$$
$$= \quad \frac{8}{\pi}\sum_{k=1}^{n}\frac{1}{k}$$

from which (4.3.3) follows immediately. ∎

Proposition 4.3.2 *Let* $V = \mathcal{C}_{\text{per}}[-\pi,\pi]$, *the space of continuous* 2π-*periodic functions with the usual sup-norm (denoted* $\|\cdot\|_\infty$*) and define* $\phi_n : V \to \mathbb{R}$ *by*

$$\phi_n(f) \;=\; s_n(f)(0)$$

where $s_n(f)$ *is the* n*-th partial sum of the Fourier series of* f. *Then* ϕ_n *is a continuous linear functional on* V *and*

$$\|\phi_n\| \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|\;dt. \tag{4.3.4}$$

Proof: On one hand,

$$\phi_n(f) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi}f(t)D_n(t)\;dt$$

(cf. (4.3.2); D_n is an even function). Thus,

$$|\phi_n(f)| \;\le\; \|f\|_\infty\frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|\;dt$$

and so

$$\|\phi_n\| \;\le\; \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|\;dt.$$

Now, let $E_n = \{t \in [-\pi,\pi] \mid D_n(t) \ge 0\}$. Define

$$f_m(t) \;=\; \frac{1 - md(t,E_n)}{1 + md(t,E_n)}$$

where $d(t,A) = \inf\{|t-s| \mid s \in A\}$ is the distance of t from a set A. Since $d(t,A)$ is a continuous function (cf. Proposition 1.2.3), $f_m \in \mathcal{C}_{\text{per}}[-\pi,\pi]$, (it is periodic since D_n is even and so E_n is a symmetric set about the origin). Also $\|f_m\|_\infty \le 1$ and $f_m(t) \to 1$ if $t \in E_n$ while $f_m(t) \to -1$ if $t \in E_n^c$. By the dominated convergence theorem, it now follows that

$$\phi_n(f_m) \;\to\; \frac{1}{2\pi}\int_{-\pi}^{\pi}|D_n(t)|\;dt$$

from which (4.3.4) follows. ■

We now apply the Banach-Steinhaus theorem to the space $V = \mathcal{C}_{\text{per}}[-\pi, \pi]$ and the collection of continuous linear functionals $\{\phi_n\}$. Since, by Propositions 4.3.1 and 4.3.2, we have $\|\phi_n\| \to \infty$ as $n \to \infty$, it follows there exists a dense G_δ-set (of continuous 2π-periodic functions) in V such that the Fourier series of all these functions diverge at $t = 0$. We could have very well dealt with any other point in the interval $[-\pi, \pi]$ in the same manner.

By another application of Baire's theorem, we can strengthen this further.

Let E_x be the dense G_δ-set of continuous 2π-periodic functions in V such that the Fourier series of these functions diverge at x. Let $\{x_i\}$ be a countable set of points in $[-\pi, \pi]$ and let

$$E = \cap_{i=1}^{n} E_{x_i} \subset V.$$

Then, by Baire's theorem, E is also a dense G_δ-set (cf. Corollary 4.1.1). Thus for each $f \in E$, the Fourier series of f diverges at x_i for all $1 \leq i \leq \infty$. Define

$$s^*(f;x) = \sup_n |s_n(f)(x)|.$$

Hence $\{x \mid s^*(f;x) = \infty\}$ is a G_δ-set in $(-\pi, \pi)$ for each f. If we choose the x_i above so that $\{x_i\}$ is dense (take all rationals, for instance in $(-\pi, \pi)$) then we have the following result.

Proposition 4.3.3 *The set $E \subset V$ is a dense G_δ-set such that for all $f \in E$, the set $Q_f \subset (-\pi, \pi)$ where its Fourier series diverges, is a dense G_δ-set in $(-\pi, \pi)$.* ■

By virtue of Corollary 4.1.2, it follows that there exist uncountably many 2π-periodic continuous functions on $[-\pi, \pi]$ whose Fourier series diverge on an uncountable subset of $(-\pi, \pi)$.

4.4 The Open Mapping and Closed Graph Theorems

In this section, we will study two more important consequences of Baire's theorem which will have a lot of applications.

We begin by setting up some notation. If A and B are subsets in a vector space V, we set

$$A + B = \{x + y \mid x \in A, y \in B\}.$$

Similarly, if λ is a scalar, we set

$$\lambda A \ = \ \{\lambda x \mid x \in A\}.$$

If A is a convex set, then we have $A + A = 2A$, for if $x \in A$, then $2x = x + x$ and so $2A \subset A + A$. On the other hand, if x and $y \in A$, by convexity, we have $\frac{1}{2}(x+y) \in A$ and so $x + y \in 2A$. Thus, $A + A \subset 2A$.

Proposition 4.4.1 *Let V and W be Banach spaces. Let $T \in \mathcal{L}(V, W)$ be surjective. Then, there exists a constant $c > 0$ such that*

$$B_W(\mathbf{0}; c) \ \subset \ T(B_V(\mathbf{0}; 1)). \tag{4.4.1}$$

where B_V and B_W denote open balls in the spaces V and W respectively.

Proof: Step 1. We will first show that there exists a constant $c > 0$ such that

$$B_W(\mathbf{0}; 2c) \ \subset \overline{T(B_V(\mathbf{0}; 1))}. \tag{4.4.2}$$

Set $X_n = n\overline{T(B_V(\mathbf{0}; 1))}$. Then each X_n is a closed set. Since T is linear and surjective, it follows immediately that $W = \cup_{n=1}^{\infty} X_n$. Hence, W being complete, it follows from Baire's theorem (cf. Remark 4.1.2) that there exists n such that X_n has non-empty interior. Hence by change of scale, it follows that

$$(\overline{T(B_V(\mathbf{0}; 1))})^\circ \ \neq \ \emptyset.$$

Thus, there exists $y_0 \in W$ and $c > 0$ such that

$$B_W(y_0; 4c) \ \subset \ \overline{T(B_V(\mathbf{0}; 1))}.$$

In particular $y_0 \in \overline{T(B_V(\mathbf{0}; 1))}$ and, by symmetry, so does $-y_0$. Since any element of $B_W(y_0; 4c)$ may be written as $y_0 + z$ where $z \in B_W(\mathbf{0}; 4c)$, any such z can be written as $z = (y_0 + z) + (-y_0)$ and so it follows from the above that

$$B_W(\mathbf{0}; 4c) \ \subset \ \overline{T(B_V(\mathbf{0}; 1))} + \overline{T(B_V(\mathbf{0}; 1))}.$$

But $\overline{T(B_V(\mathbf{0}; 1))}$ is convex and so the set on the right-hand side in the above inclusion is, in fact, $2\overline{T(B_V(\mathbf{0}; 1))}$ and so we have (4.4.2).

Step 2. We now prove (4.4.1). Let $y \in B_W(\mathbf{0}; c)$. We need to find $x \in B_V(\mathbf{0}; 1)$ such that $T(x) = y$. Let $\varepsilon > 0$. There exists $z \in V$ such that $\|z\|_V < 1/2$ and $\|y - T(z)\|_W < \varepsilon$ by virtue of (4.4.2) (applied to $2y$). Set $\varepsilon = c/2$ and let $z_1 \in V$ be such that $\|z_1\|_V < 1/2$ and $\|T(z_1) - y\|_W < c/2$.

We can iterate this procedure. By another application of (4.4.2) (to $4(T(z_1) - y)$) we can find $z_2 \in V$ such that

$$\|z_2\|_V \ < \ \frac{1}{4}, \ \|T(z_1 + z_2) - y\|_W \ < \ \frac{c}{4}.$$

Thus, we can find, by repeated use of (4.4.2), a sequence $\{z_n\}$ in V such that

$$\|z_n\|_V \ < \ \frac{1}{2^n}, \ \|T(z_1 + \cdots + z_n) - y\|_W \ < \ \frac{c}{2^n}.$$

Then, it follows that the sequence $\{z_1 + \cdots + z_n\}$ is Cauchy in V, and, since V is complete, it will converge to an element $z \in V$ such that $\|z\|_V < 1$ and we will also have $T(z) = y$. This completes the proof. ■

Theorem 4.4.1 *(***Open Mapping Theorem***) Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be* surjective. *Then T is an open map.*

Proof: We need to show that T maps open sets in V onto open sets in W. Let G be an open set in V. Let $y \in T(G)$. Then, there exists $x \in G$ such that $y = T(x)$. Since G is open, there exists $r > 0$ such that $x + B_V(\mathbf{0}; r) \subset G$. Hence, $y + T(B_V(\mathbf{0}; r)) \subset T(G)$. But by the previous proposition, there exists $s > 0$ such that $B_W(\mathbf{0}; s) \subset T(B_V(\mathbf{0}; r))$ and so $y + B_W(\mathbf{0}; s) \subset T(G)$ which means that $T(G)$ is open. ■

Corollary 4.4.1 *Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be a bijection. Then T is an isomorphism.*

Proof: Since T is a continuous bijection, in particular it is onto and so, by the open mapping theorem, it is an open map, which means that T^{-1} is also continuous. Thus T is an isomorphism. ■

Corollary 4.4.2 *Let V be a Banach space with respect to two norms, $\|.\|_1$ and $\|.\|_2$. Assume that there exists $c > 0$ such that*

$$\|x\|_1 \ \leq \ c\|x\|_2$$

for all $x \in V$. Then, the two norms are equivalent.

Proof: The identity map $I : \{V, \|.\|_2\} \rightarrow \{V, \|.\|_1\}$ is a linear bijection which is also continuous by the given hypothesis. Hence the inverse map is also continuous and this means that there exists a constant $C > 0$ such that

$$\|x\|_2 \ \leq \ C\|x\|_1$$

for all $x \in V$. Hence the norms are equivalent. ■

If V and W are normed linear spaces and $T : V \to W$ is any mapping define the **graph** of T, denoted $G(T)$ as follows:

$$G(T) \;=\; \{(x, y) \mid y = T(x)\} \;\subset\; V \times W$$

If V and W are normed linear spaces and $T : V \to W$ is continuous, then $G(T)$ is closed in $V \times W$. This follows from the following general topological result.

Lemma 4.4.1 *Let X and Y be topological spaces, with Y being Hausdorff. Let $f : X \to Y$ be a given mapping. Then, if f is continuous, the graph*

$$G(f) \;=\; \{(x, y) \in X \times Y \mid y = f(x)\}$$

is closed in the product space $X \times Y$.

Proof: Let $(x, y) \in (X \times Y)\backslash G(f)$. Thus, $y \neq f(x)$ and so, since Y is Hausdorff, there exist open sets $\mathcal{U}$ and $\mathcal{V}$ in Y such that $f(x) \in \mathcal{U}, y \in \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \emptyset$. Now, $f^{-1}(\mathcal{U}) \times \mathcal{V}$ is open in $X \times Y$ and contains the point (x, y). Further, this set does not intersect $G(f)$. (For, if (u, v) were in the intersection, we would have $v = f(u), u \in f^{-1}(\mathcal{U})$ and $v \in \mathcal{V}$. Thus, $f(u) = v \in \mathcal{U} \cap \mathcal{V}$, which is a contradiction.) Thus, the complement of $G(f)$ contains an open neighbourhood of each of its points and is hence open, *i.e.* $G(f)$ is closed in $X \times Y$. ■

If V and W are Banach spaces and T is linear, then the converse is also true as the following theorem shows.

Theorem 4.4.2 *(***Closed Graph Theorem***) Let V and W be Banach spaces and let $T : V \to W$ be a linear mapping. Assume that $G(T)$, the graph of T, is closed in $V \times W$. Then T is continuous.*

Proof: For $x \in V$, define

$$\|x\|_1 \;=\; \|x\|_V + \|T(x)\|_W.$$

Then $\|.\|_1$ defines a norm on V. If $\{x_n\}$ is a sequence in V which is Cauchy with respect to this norm, then, evidently it is Cauchy with respect to the norm $\|.\|_V$ as well. Then, since V is complete for the norm $\|.\|_V$, we have that $x_n \to x$ in V (in the topology of the norm $\|.\|_V$). Again, since $\{x_n\}$ is Cauchy for the norm $\|.\|_1$, it also follows that $\{T(x_n)\}$ is Cauchy in W and so, since W is complete, $T(x_n) \to y$ in W. Since $G(T)$ is closed, it follows that $y = T(x)$. Thus, it follows that $x_n \to x$ in the topology of the norm $\|.\|_1$ as well. Thus $\{V, \|.\|_1\}$ is also complete. Hence, by the preceding corollary, $\|.\|_V$ and $\|.\|_1$ are equivalent. Hence, there exists $C > 0$ such that $\|x\|_1 \leq C\|x\|_V$ for all $x \in V$. In particular, $\|T(x)\|_W \leq C\|x\|_V$ for all $x \in V$ and so T is continuous. ■

Remark 4.4.1 The closed graph theorem gives a convenient way for verifying the continuity of linear maps between Banach spaces V and W. We just have to verify that if $x_n \to x$ in V and if $T(x_n) \to y$ in W, then $y = T(x)$. ■

Remark 4.4.2 In the Banach-Steinhaus theorem, it was sufficient that the domain space V was complete. The range space W could be any normed linear space. In the case of the Open Mapping and Closed Graph theorems, however, it is essential that both the spaces V and W are complete. ■

4.5 Annihilators

In the remaining sections of this chapter, we will apply the open mapping theorem to obtain results about the range and the kernel of linear maps. In order to do this, we need to introduce an important notion.

Definition 4.5.1 *Let V be a Banach space and let W be a subspace of V. Let Z be a subspace of V^*. The* **annihilator** *of W is the subspace of V^* given by*

$$W^\perp = \{f \in V^* \mid f(x) = 0 \text{ for all } x \in W\}.$$

The annihilator of Z is the subspace of V given by

$$Z^\perp = \{x \in V \mid f(x) = 0 \text{ for all } f \in Z\}. \blacksquare$$

It is easy to see that $W^\perp$ is a closed subspace of V^* and that $Z^\perp$ is a closed subspace of V. Further,

$$\left(W^\perp\right)^\perp = \overline{W} \tag{4.5.1}$$

and

$$\left(Z^\perp\right)^\perp \supset \overline{Z} \tag{4.5.2}$$

(cf. Exercises 3.8 and 3.9). Also, if G and H are subspaces of V such that $G \subset H$, then $H^\perp \subset G^\perp$.

Proposition 4.5.1 *Let G and H be closed subspaces of a Banach space V. Then*

$$G \cap H = \left(G^\perp + H^\perp\right)^\perp \tag{4.5.3}$$

and

$$G^\perp \cap H^\perp = (G + H)^\perp. \tag{4.5.4}$$

Proof: Clearly, $G \cap H \subset (G^\perp + H^\perp)^\perp$. Also, $G^\perp \subset G^\perp + H^\perp$ and $H^\perp \subset G^\perp + H^\perp$. Hence,

$$\left(G^\perp + H^\perp\right)^\perp \subset \left(G^\perp\right)^\perp = \overline{G} = G$$

using (4.5.1) and the fact that G is closed. Similarly, we have

$$\left(G^\perp + H^\perp\right)^\perp \subset H$$

and (4.5.3) follows immediately. The relation (4.5.4) is obvious. ■

Combining the result of the preceding proposition with the relations (4.5.1)-(4.5.2), we deduce the following corollary.

Corollary 4.5.1 *Let G and H be closed subspaces of a Banach space V. Then*

$$(G \cap H)^\perp \supset \overline{G^\perp + H^\perp} \tag{4.5.5}$$

and

$$\left(G^\perp \cap H^\perp\right)^\perp = \overline{G + H}. \blacksquare \tag{4.5.6}$$

Proposition 4.5.2 *Let G and H be two closed subspaces of a Banach space V such that $G + H$ is also closed. Then, there exists a constant $C > 0$ such that, for every $z \in G + H$, there exist $x \in G$ and $y \in H$ satisfying*

$$z = x + y, \ \|x\| \leq C\|z\|, \ \|y\| \leq C\|z\|. \tag{4.5.7}$$

Proof: Consider the product space $G \times H$ with the norm

$$\|(x, y)\|_{G \times H} = \|x\| + \|y\|.$$

This is a Banach space since G and H are closed. Consider the map $G \times H \to G+H$ given by $(x,y) \mapsto x+y$. This map is clearly continuous, linear and onto. Since $G+H$ is a closed subspace of the Banach space V, it is also a Banach space and so, by the open mapping theorem, there exists a constant $c > 0$ such that if $z \in G+H$ with $\|z\| < c$, then there exists $(x,y) \in G \times H$ such that $z = x+y$ and such that $\|x\| + \|y\| < 1$. Given any $z \in G+H$, by considering, for instance, the vector $\frac{c}{2\|z\|}z$ we see immediately that z can be written as $z = x+y$ with $x \in G$, $y \in H$ such that

$$\|x\| \ \le \ \frac{2}{c}\|z\|, \ \|y\| \ \le \ \frac{2}{c}\|z\|$$

which proves the result with $C = 2/c$. ■

Corollary 4.5.2 *Let G and H be closed subspaces of a Banach space V such that $G+H$ is also closed. Then, there exists a constant $C > 0$ such that*

$$d(x, G\cap H) \ \le \ C\,[d(x,G) + d(x,H)] \tag{4.5.8}$$

for all $x \in V$.

Proof: Let $x \in V$ and let $\varepsilon > 0$ be arbitrary. Then, there exist $a \in G$ and $b \in H$ such that

$$\|x-a\| \ \le \ d(x,G)+\varepsilon \ \text{ and } \ \|x-b\| \ \le \ d(x,H)+\varepsilon.$$

set $z = a-b \in G+H$. Then, there exists $c > 0$ (which depends only on G and H), $a' \in G$ and $b' \in H$ such that

$$a-b \ = \ a'+b', \ \|a'\| \ \le \ c\|a-b\|, \ \|b'\| \ \le \ c\|a-b\|.$$

Now, $a - a' = b' + b \in G\cap H$. Thus,

$$\begin{aligned}
d(x, G\cap H) \ &\le \ \|x-a\| + \|a'\| \\
&\le \ \|x-a\| + c\|a-b\| \\
&\le \ (1+c)\|x-a\| + c\|x-b\| \\
&\le \ (1+c)d(x,G) + cd(x,H) + (1+2c)\varepsilon \\
&\le \ (1+c)(d(x,G) + d(x,H)) + (1+2c)\varepsilon
\end{aligned}$$

which completes the proof (with $C = (1+c)$), since ε is an arbitrarily small quantity. ■

We are now in a position to prove a deeper result on annihilators.

Theorem 4.5.1 *Let G and H be closed subspaces of a Banach space V. The following are equivalent:*
(i) $G+H$ *is closed in* V.
(ii) $G^\perp + H^\perp$ *is closed in* V^*.
(iii)
$$G+H = \left(G^\perp \cap H^\perp\right)^\perp.$$
(iv)
$$G^\perp + H^\perp = (G\cap H)^\perp.$$

Proof: Since an annihilator is always closed, the implication (iv) $\Rightarrow$ (ii) is trivial. The equivalence (i) $\Leftrightarrow$ (iii) is an immediate consequence of (4.5.6). To complete the proof, we need to show that (i) $\Rightarrow$ (iv) and that (ii) $\Rightarrow$ (i), which we now proceed to do.

Step 1. (i) $\Rightarrow$ (iv). Assume that $G+H$ is closed. Now, by (4.5.5), we already know that $(G\cap H)^\perp \supset G^\perp + H^\perp$. Hence, to prove (iv), it suffices to prove the reverse inclusion. Let $f \in (G\cap H)^\perp$. Define a linear functional φ on $G+H$ as follows: let $x = a+b \in G+H$ with $a\in G$ and $b \in H$; set
$$\varphi(x) \;=\; f(a).$$
If $x = a'+b'$ is another decomposition of x with $a' \in G$ and $b' \in H$, it follows that $a - a' = b' - b \in G\cap H$ and so $f(a) = f(a')$. Thus the definition of $\varphi(x)$ is independent of the decomposition chosen, and so φ is a well defined linear functional. Further, since $G+H$ is closed, we can choose a decomposition $x = a+b$ such that $\|a\| \leq C\|x\|$ (where C depends only on G and H). Consequently, $\|\varphi(x)\| \leq C\|f\|\,\|x\|$ and it follows that φ is a continuous linear functional on $G+H$. Hence, we can extend it to a continuous linear functional $\widetilde{\varphi}$ on V, by the Hahn-Banach theorem. Now, $f = \varphi = \widetilde{\varphi}$ on G and so $f - \widetilde{\varphi} \in G^\perp$. Also if $x \in H$, then since $\varphi(x) = 0$, it follows that $\widetilde{\varphi} \in H^\perp$. Hence we have that
$$f \;=\; (f-\widetilde{\varphi}) + \widetilde{\varphi} \;\in\; G^\perp + H^\perp$$
which proves the reverse inclusion that we sought to establish.

Step 2. (ii) $\Rightarrow$ (i). For any $f \in V^*$, we have that (cf. Exercise (3.8)(b))
$$d(f, G^\perp) \;=\; \|f|_G\| \;=\; \sup_{x\in G,\ \|x\|\leq 1} |f(x)|,$$

$$d(f, H^{\perp}) = \|f|_H\| = \sup_{x \in H, \ \|x\| \le 1} |f(x)|,$$

and (in view of (4.5.4)) that

$$d(f, G^{\perp} \cap H^{\perp}) = d(f, (G+H)^{\perp}) = \|f|_{\overline{G+H}}\| = \sup_{x \in G+H, \ \|x\| \le 1} |f(x)|.$$

By Corollary 4.5.2, there exists a constant $C > 0$ such that, for every $f \in V^*$,

$$d(f, G^{\perp} \cap H^{\perp}) \le C \left[d(f, G^{\perp}) + d(f, H^{\perp}) \right]$$

since we are assuming that $G^{\perp} + H^{\perp}$ is closed in V^*. In other words, we have, for every $f \in V^*$,

$$\sup_{x \in \overline{G+H}, \ \|x\| \le 1} |f(x)| \le C \left[\sup_{x \in G, \ \|x\| \le 1} |f(x)| + \sup_{x \in H, \ \|x\| \le 1} |f(x)| \right]. \tag{4.5.9}$$

Step 3. We now claim that the above relation implies that

$$\overline{B_G(\mathbf{0}; 1) + B_H(\mathbf{0}; 1)} \supset \frac{1}{C} B_{\overline{G+H}}(\mathbf{0}; 1) \tag{4.5.10}$$

where $B_G(\mathbf{0}; 1)$ denotes the (open) unit ball in G and so on. If not, let $x_0 \in \overline{G+H}$ with $\|x_0\| < 1/C$ such that

$$x_0 \notin \overline{B_G(\mathbf{0}; 1) + B_H(\mathbf{0}; 1)}.$$

Assume that V is a real Banach space. Then, by the Hahn-Banach theorem, there exists $f \in V^*$ and a real number α such that, for all $z \in B_G(\mathbf{0}; 1) + B_H(\mathbf{0}; 1)$, we have

$$f(z) < \alpha < f(x_0). \tag{4.5.11}$$

In particular, $f(x_0) > \alpha > 0$. Further, if $z = x + y \in B_G(\mathbf{0}; 1) + B_H(\mathbf{0}; 1)$, then so does $-z = -x - y$. Thus

$$\begin{aligned} \sup_{x \in G, \ \|x\| \le 1} |f(x)| + \sup_{x \in H, \ \|x\| \le 1} |f(x)| &\le \alpha \\ &< f(x_0) \\ &= \|x_0\| f\left(\tfrac{1}{\|x_0\|} x_0\right) \\ &< \tfrac{1}{C} f\left(\tfrac{1}{\|x_0\|} x_0\right) \\ &\le \tfrac{1}{C} \sup_{\substack{x \in \overline{G+H} \\ \|x\| \le 1}} |f(x)| \end{aligned}$$

which contradicts (4.5.9). This establishes (4.5.10).

In case V is a complex Banach space, then (4.5.11) holds for the real part of a continuous linear functional f and the preceding sequence of inequalities hold with $|f(x)|$ being replaced by $|\mathrm{Re}(f)(x)|$. However (cf. Proposition 3.1.1) since the norm $\|f\|$ is the same as the norm $\|\mathrm{Re}(f)\|$ (the latter being considered as a real linear functional), and since the supremum over the unit ball gives the norm of the functional concerned, the same conclusion holds with $|f(x)|$ as well and so we again get a contradiction to (4.5.9).

Step 4. Now consider the spaces $E = G \times H$ with the norm $\|(x,y)\|_E = \max\{\|x\|, \|y\|\}$ and $F = \overline{G+H}$ with the norm from V. Both are Banach spaces. Define $T : E \to F$ by $T((x,y)) = x + y$. Then T is continuous and linear. Further, in view of (4.5.10), we also have that

$$\overline{T\left(B_E(\mathbf{0};1)\right)} \supset \frac{1}{C} B_F(\mathbf{0};1).$$

Now, this implies that (cf. Step 2 of the proof of Proposition 4.4.1)

$$T\left(B_E(\mathbf{0};1)\right) \supset \frac{1}{2C} B_F(\mathbf{0};1).$$

But then, it is now immediate to see that T must be onto. In other words, $\overline{G+H} = G+H$, *i.e.* $G+H$ is closed. This completes the proof. ■

4.6 Complemented Subspaces

Let V be a vector space. If W is a subspace of V, then by completing a basis of W to get a basis of V, we can easily produce a subspace Z of V such that $V = W \oplus Z$. The question which we wish to address now is that if W is a *closed* subspace of a Banach space V, whether there exists a *closed* subspace Z as above.

Definition 4.6.1 *Let G be a closed subspace of a Banach space V. A* closed *subspace H of V is said to be a* **complement** *of G if $V = G \oplus H$, i.e. $G \cap H = \{\mathbf{0}\}$ and $V = G + H$.* ■

Remark 4.6.1 If G has a complement H in V, then every $x \in V$ has a unique decomposition $z = x + y$ with $x \in G$ and $y \in H$. By Proposition 4.5.2, it follows that the maps $z \mapsto x$ and $z \mapsto y$ are continuous. Thus, we have continuous projections from V onto G and H. If G and H were not closed, these projections need not be continuous. ■

Example 4.6.1 Let G be a finite dimensional subspace of a Banach space V. Then G has a complement (cf. Exercises 3.5 and 3.6). ■

Example 4.6.2 Every closed subspace G with finite codimension has a complement. This is trivial since any algebraic complement, *i.e.* any subspace H such that $V = G \oplus H$, is finite dimensional and is hence closed. ■

Remark 4.6.2 Subspaces of finite codimension typically occur in the following way. Let Z be a subspace of V^* of dimension d. Then its annihilator $Z^\perp$ will be a subspace of V with codimension d. To see this, let $\{f_1, \cdots, f_d\}$ be a basis for Z. Define $\varphi : V \to \mathbb{R}^d$ by $\varphi(x) = (f_1(x), \cdots, f_d(x))$. This map is onto. If not, by the Hahn-Banach theorem, there exist scalars $\{\alpha_1, \cdots, \alpha_d\}$, not all zero, such that

$$\sum_{i=1}^{d} \alpha_i f_i(x) \;=\; 0$$

for every $x \in V$. But this implies that $\sum_{i=1}^{d} \alpha_i f_i = \mathbf{0}$ in V^*, which contradicts the linear independence of the $\{f_i\}$. Thus, we can find vectors $\{e_1, \cdots, e_d\}$ in V such that

$$f_i(e_j) \;=\; \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

It is now easy to verify that $\{e_1, \cdots, e_d\}$ are linearly independent and that their span (which has dimension d) is a complement to $Z^\perp$. ■

Remark 4.6.3 We will see later (cf. Chapter 7) that in a *Hilbert space*, every closed subspace is complemented. In fact, a deep result of Lindenstrauss and Tzafriri states that any Banach space which is not isomorphic to a Hilbert space will always have uncomplemented closed subspaces. Since Hilbert spaces are always reflexive, it follows that, in particular, non-reflexive spaces will always have uncomplemented closed subspaces. For example de Vito has shown that the space c_0 (cf. Exercise 3.7) is uncomplemented in ℓ_∞. ■

Definition 4.6.2 *Let V and W be Banach spaces and let $T \in \mathcal{L}(V,W)$ be surjective. A linear transformation $S \in \mathcal{L}(W,V)$ is said to be a* **right inverse** *of T if $T \circ S = I_W$, the identity operator on W.* ■

Proposition 4.6.1 *Let V and W be Banach spaces and let $T \in \mathcal{L}(V,W)$ be surjective. The following are equivalent:*
(i) *T has a right inverse.*
(ii) *The subspace $N = \mathrm{Ker}(T)$ is complemented in V.*

Proof: Step 1. (i) $\Rightarrow$ (ii). Let $S \in \mathcal{L}(W,V)$ be a right inverse for T. Consider the image $S(W) \subset V$ of S. Clearly, $S(W) \cap N = \{\mathbf{0}\}$. Also, if $x \in V$, then $x - S(T(x)) \in N$. Thus $V = S(W) \oplus N$. We now show that $S(W)$ is closed. Indeed, if we have a sequence $\{y_n\}$ in W such that $S(y_n) \to x$ in V, then $y_n = T(S(y_n)) \to T(x)$ in W. Then, it follows that

$$x \;=\; \lim_{n\to\infty} S(y_n) \;=\; S(T(x))$$

i.e. $x \in S(W)$, which proves that $S(W)$ is closed. Thus N is complemented in V.

Step 2. (ii) $\Rightarrow$ (i). Assume that a closed subspace M of V is a complement to N in V. Then, we have a continuous projection $P : V \to M$. Let $y \in W$. Let $x \in V$ such that $T(x) = y$ (T is surjective). If $x' \in V$ also satisfies $T(x') = y$, then $x - x' \in N$ and so $P(x) = P(x')$. Thus, the map $y \mapsto P(x)$ is well defined. Define $S(y) = P(x)$. Then $T(S(y)) = T(P(x)) = T(x) = y$ (since $x - P(x) \in N$ and so $T(x) = T(P(x))$). Thus $T \circ S = I_W$. We now show that S is continuous as well. Since T is onto, there exists a constant $C > 0$ such that $B_W(\mathbf{0}; C) \subset T(B_V(\mathbf{0}; 1))$. Thus, for any $y \in W$, $\frac{C}{2\|y\|_W} y \in B_W(\mathbf{0}; C)$ and so there exists $x' \in V$ with $\|x'\|_V < 1$ such that

$$T(x') \;=\; \frac{C}{2\|y\|_W} y.$$

Hence, if we set $x = \frac{2}{C}\|y\|_W x'$, we get that $T(x) = y$ and that

$$\|x\|_V \;<\; \frac{2}{C}\|y\|_W.$$

Thus,

$$\|S(y)\|_V \;=\; \|P(x)\|_V \;\le\; \|P\|\|x\|_V \;\le\; \frac{2}{C}\|P\|\;\|y\|_W$$

which proves the continuity of S. Thus S is a right inverse for T. ■

4.7 Unbounded Operators, Adjoints

In this section, we will look at linear transformations between Banach spaces which are not necessarily defined on all of the space, but only on a subspace. Further, they may not map bounded sets into bounded sets. Such transformations are said to be unbounded and bounded linear transformations form a special subclass.

Definition 4.7.1 *Let V and W be Banach spaces. An* **unbounded linear operator** *(or,* **transformation***) from V into W is any linear map defined on a subspace of V taking values in W. The domain of definition of the transformation A is called the* **domain** *of A and is denoted $D(A)$. Thus,*

$$A : D(A) \subset V \to W.$$

The image of A is a subspace of W and is called the **range** *of A and is denoted $\mathcal{R}(A)$. The operator A is said to be* **bounded** *if there exists a constant $C > 0$ such that*

$$\|A(x)\|_W \ \leq \ C\|x\|_V \tag{4.7.1}$$

for all $x \in D(A)$. The operator A is said to be **densely defined** *if $\overline{D(A)} = V$. The* **graph** *of a linear operator A is denoted $G(A)$ and is given by*

$$G(A) \ = \ \{(x, A(x)) \in V \times W \mid x \in D(A)\}.$$

The operator A is said to be **closed** *if the graph $G(A)$ is closed in $V \times W$.* ■

To define a linear operator, we thus need to specify its domain and then its action on vectors in the domain. Given a linear operator $A : D(A) \subset V \to W$, we denote its kernel by $\mathcal{N}(A)$. Thus,

$$\mathcal{N}(A) \ = \ \{x \in D(A) \mid A(x) = \mathbf{0}\}.$$

Remark 4.7.1 If $A : D(A) \subset V \to W$ is closed, then $\mathcal{N}(A)$ is a closed subspace in V. ■

Example 4.7.1 Let $V = W = \mathcal{C}[0,1]$. Let $D(A) = \mathcal{C}^1[0,1]$. Define $A : D(A) \subset V \to V$ by $A(u) = u'$ where u' stands for the derivative of the function u. Clearly, A is densely defined and $\mathcal{N}(A)$ is the subspace of all constant functions. It follows from the fundamental theorem of calculus that A is surjective. If $u_n \to u$ uniformly and if $u'_n \to v$ uniformly, we know that u is differentiable and that $u' = v$. Thus, A is a closed operator. Finally, A is unbounded. This, in fact, is the content of Example 2.3.8. ■

Notation: Let V be a Banach space. Let $x \in V$ and $f \in V^*$. We introduce the **duality bracket** $< f, x >$, which will also be denoted $< f, x >_{V^*,V}$ in case we need to specify the spaces involved, via the relation

$$< f, x > = f(x).$$

Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a *densely defined* linear operator. Let

$$Z = \left\{ v \in W^* \mid \begin{array}{c} \text{there exists } C > 0 \text{ such that for all } x \in D(A) \\ |< v, A(x) >_{W^*,W}| \le C\|x\|_V \end{array} \right\}. \tag{4.7.2}$$

Note that the constant C mentioned above depends on v. Clearly, Z is a subspace of W^*. If $v \in Z$, define, for $x \in D(A)$,

$$g(x) = < v, A(x) >_{W^*,W}.$$

By the definition of Z, it follows that g defines a continuous linear functional on $D(A)$. Hence, by the Hahn-Banach theorem, we can extend it to a continuous linear functional g_v on all of V. Since $D(A)$ is dense in V, such an extension is unique.

To summarize, we have a map $v \mapsto g_v$ from Z into V^*. It is also easy to see that this map is linear.

Definition 4.7.2 *Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a* densely defined *linear operator. Let Z be as defined above and for each $v \in Z$, let $g_v \in V^*$ be as defined above. We set $D(A^*) = Z$ and define $A^*(v) = g_v$ for $v \in D(A^*)$. The linear operator $A^* : D(A^*) \subset W^* \to V^*$ is called the* **adjoint** *of the operator A.* ■

Thus, the adjoint is defined for densely defined linear operators. Notice that there is no reason for A^* to be densely defined. We have the following important duality relationship: for all $u \in D(A)$ and all $v \in D(A^*)$, we have

$$< A^*(v), u >_{V^*,V} = < v, A(u) >_{W^*,W}. \tag{4.7.3}$$

Example 4.7.2 In a finite dimensional space, every subspace is closed. Thus, any linear transformation is closed; it is densely defined if, and only if, it is defined over the entire space. Hence, every linear transformation defined on the entire space has an adjoint and, since every linear transformation is bounded, we also have that the adjoint is defined on the entire dual space of the range. The dual of $\mathbb{C}^n$ is identified with $\mathbb{C}^n$ with the duality product being given by

$$< y, x > = \sum_{i=1}^{n} x_i \overline{y_i}$$

where $x = (x_1, \cdots, x_n)$ is in the base space and $y = (y_1, \cdots, y_n)$ is in the dual space. If $A : \mathbb{C}^n \to \mathbb{C}^m$ is a linear transformation which is represented by the $m \times n$ matrix $\mathbf{A}$, then we have, for all $\mathbf{y} \in \mathbb{C}^m$ and for all $\mathbf{x} \in \mathbb{C}^n$,

$$< A^* y, x > = < y, Ax > = \sum_{i=1}^{m} \left(\sum_{j=1}^{n} a_{ij} x_j \right) \overline{y_i} = \sum_{j=1}^{n} x_j \overline{\left(\sum_{i=1}^{m} \overline{a_{ij}} y_i \right)}$$

from which it follows that the matrix representing A^* is none other than the adjoint matrix $\mathbf{A}^*$ (cf. Definition 1.1.11).

In the same way, if the matrix $\mathbf{A}$ represents a linear transformation A from $\mathbb{R}^n$ into $\mathbb{R}^m$, the adjoint of this transformation will be represented by the transpose matrix $\mathbf{A}'$. ■

Example 4.7.3 Consider the space ℓ_2 of square summable real sequences. Then its dual space can be identified with itself (cf. Example 3.1.1). Consider the linear operators on ℓ_2 given by $T(x) = (0, x_1, x_2, \cdots)$ and $S(x) = (x_2, x_3, \cdots)$ where $x = (x_1, x_2, \cdots) \in \ell_2$. Then it is easy to check that $T^* = S$ and that $S^* = T$. ■

Proposition 4.7.1 *Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a densely defined linear operator. Then A^* is closed.*

Proof: We need to show that $G(A^*)$, the graph of A^*, is closed in $W^* \times V^*$. Let $v_n \to v$ in W^* and let $A^*(v_n) \to f$ in V^*. Let $u \in D(A)$. Then

$$< A^*(v_n), u >_{V^*,V} = < v_n, A(u) >_{W^*,W} .$$

Passing to the limit as $n \to \infty$, we get

$$< f, u >_{V^*,V} = < v, A(u) >_{W^*,W} \tag{4.7.4}$$

for all $u \in D(A)$. Thus,

$$| < v, A(u) >_{W^*,W} | \leq \|f\|_{V^*} \|u\|_V$$

which shows that $v \in D(A^*)$. It also follows from (4.7.4) that $f = A^*(v)$, thus proving that $G(A^*)$ is closed. ■

The graphs of A and A^* are connected by a simple relation. Let V and W be Banach spaces. Define

$$\mathcal{J} : W^* \times V^* \to V^* \times W^*$$

by

$$\mathcal{J}(v, f) = (-f, v).$$

Proposition 4.7.2 *Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a densely defined linear operator. Let $\mathcal{J}$ be as defined above. Then*

$$\mathcal{J}(G(A^*)) = (G(A))^\perp.$$

Proof: Let $u \in D(A)$ be an arbitrary element so that $(u, A(u))$ is an arbitrary element of $G(A)$.

$$\begin{aligned} (v, f) \in G(A^*) &\Leftrightarrow \; < f, u >_{V^*,V} = < v, A(u) >_{W^*,W} \\ &\Leftrightarrow \; < -f, u >_{V^*,V} + < v, A(u) >_{W^*,W} = 0 \\ &\Leftrightarrow \; \mathcal{J}(v, f) \in (G(A))^\perp \end{aligned}$$

which completes the proof. ■

The following result characterizes densely defined and closed operators that are bounded.

Proposition 4.7.3 *Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a densely defined and closed linear operator. The following are equivalent:*
(i) $D(A) = V$.
(ii) *A is bounded.*
(iii) $D(A^*) = W^*$.
(iv) *A^* is bounded.*
In this case, we also have

$$\|A\| = \|A^*\|. \tag{4.7.5}$$

Proof: (i) $\Rightarrow$ (ii). If $D(A) = V$ and A is closed, then it is continuous by the closed graph theorem, and hence is bounded.

(ii) $\Rightarrow$ (iii). If A is bounded, then it follows from the definition of A^* that $D(A^*) = W^*$.

(iii) $\Rightarrow$ (iv). Since $G(A^*)$ is always closed, the result again follows from the closed graph theorem.

(iv) $\Rightarrow$ (i). First of all, we show that $D(A^*)$ is closed. Indeed, if $\{v_n\}$ is a sequence in $D(A^*)$ converging to $v \in W^*$, then,

$$\|A^*(v_n - v_m)\| \ \leq C\|v_n - v_m\|$$

since A^* is bounded. Thus, it follows that $\{A^*(v_n)\}$ is Cauchy in V^*. Let $A^*(v_n) \to f$ in V^*. Since $G(A^*)$ is always closed, it follows then that $v \in D(A^*)$ and that $A^*(v) = f$. Thus $D(A^*)$ is closed.

Now set $G = G(A) \subset V \times W$ and $H = \{\mathbf{0}\} \times W \subset V \times W$. Both are closed subspaces. Further, $G + H = D(A) \times W$. On the other hand, since $(G(A))^\perp = \mathcal{J}(G(A^*))$, we have $G^\perp + H^\perp = V^* \times D(A^*)$, which is closed. Thus, by Theorem 4.5.1, it follows that $G + H$ is closed as well, which implies that $D(A)$ is closed. Thus, $D(A) = \overline{D(A)} = V$.

This proves the equivalence of all the four statements. Under these conditions, we now prove (4.7.5). For all $u \in V$ and for all $v \in W^*$, we have

$$< v, A(u) >_{W^*,W} = < A^*(v), u >_{V^*,V} \tag{4.7.6}$$

which yields

$$| < v, A(u) >_{W^*,W} | \ \leq \ \|A^*\|\|v\|_{W^*}\|u\|_V$$

whence we deduce that (cf. Corollary 3.1.2)

$$\|A(u)\|_W \ \leq \ \|A^*\| \ \|u\|_V$$

which implies that $\|A\| \leq \|A^*\|$.

Again, by virtue of (4.7.6), it follows that

$$\|A^*(v)\|_{V^*} \ = \ \sup_{u \in V, \ \|u\| \leq 1} | < A^*(v), u >_{V^*,V} | \ \leq \ \|A\| \ \|v\|_{W^*}$$

which implies that $\|A^*\| \leq \|A\|$. This completes the proof. ■

Proposition 4.7.4 *Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a densely defined linear operator. Let $G = G(A) \subset V \times W$ and let $H = V \times \{\mathbf{0}\} \subset V \times W$. Then*
(i) $\mathcal{N}(A) \times \{\mathbf{0}\} = G \cap H$.
(ii) $V \times \mathcal{R}(A) = G + H$.
(iii) $\{\mathbf{0}\} \times \mathcal{N}(A^*) = G^\perp \cap H^\perp$.
(iv) $\mathcal{R}(A^*) \times W^* = G^\perp + H^\perp$.

Proof: The proof is an immediate consequence of the definitions and the relation $G(A)^\perp = \mathcal{J}(G(A^*))$. The details are left as an exercise. ■

Corollary 4.7.1 *Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a* closed *and* densely defined *linear operator. Then*

$$\mathcal{N}(A) = \mathcal{R}(A^*)^\perp, \tag{4.7.7}$$

$$\mathcal{N}(A^*) = \mathcal{R}(A)^\perp, \tag{4.7.8}$$

$$\mathcal{N}(A)^\perp \supset \overline{\mathcal{R}(A^*)} \tag{4.7.9}$$

and

$$\mathcal{N}(A^*)^\perp = \overline{\mathcal{R}(A)}. \tag{4.7.10}$$

Proof: Let G and H be as in the preceding proposition. We know that (cf. Proposition 4.5.1) $G \cap H = (G^\perp + H^\perp)^\perp$. Thus, we get that

$$\mathcal{N}(A) \times \{\mathbf{0}\} = \mathcal{R}(A^*)^\perp \times \{\mathbf{0}\}.$$

This proves (4.7.7). Again, by Proposition 4.5.1, we know that $G^\perp \cap H^\perp = (G + H)^\perp$ which yields

$$\{\mathbf{0}\} \times \mathcal{N}(A^*) = \{\mathbf{0}\} \times \mathcal{R}(A)^\perp$$

which proves (4.7.8). The remaining two relations are proved from these two and applying (4.5.1) and (4.5.2). ■

Theorem 4.7.1 *Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a* closed *and* densely defined *linear operator. Then, the following are equivalent:*
(i) *$\mathcal{R}(A)$ is closed in W.*
(ii) *$\mathcal{R}(A^*)$ is closed in V^*.*
(iii) $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$.
(iv) $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$.

Proof: Using the same notations as in the preceding propoiition and its corollary, we have:
(i) $\Leftrightarrow G+H$ is closed in $V \times W$.
(ii) $\Leftrightarrow G^\perp + H^\perp$ is closed in $V^* \times W^*$.
(iii) $\Leftrightarrow G+H = (G^\perp \cap H^\perp)^\perp$.
(iv) $\Leftrightarrow G^\perp + H^\perp = (G \cap H)^\perp$.

The equivalence of these statements follows from Theorem 4.5.1. ■

Remark 4.7.2 As already remarked (cf. Example 4.7.2), all linear transformations on finite dimensional spaces are closed and densely defined means defined on all the space. The statement that $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp$ is none other than the well known Fredhölm alternative (cf. Proposition 1.1.5). ■

We conclude with a useful characterization of surjective maps.

Theorem 4.7.2 *Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a* closed *and* densely defined *linear operator. The following are equivalent:*
(i) *A is onto, i.e. $\mathcal{R}(A) = W$.*
(ii) *There exists a constant $C > 0$ such that, for all $v \in D(A^*)$,*

$$\|v\|_{W^*} \leq C\|A^*(v)\|_{V^*}. \tag{4.7.11}$$

(iii) *$\mathcal{N}(A^*) = \{\mathbf{0}\}$ and $\mathcal{R}(A^*)$ is closed in V^*.*

Proof: (i) $\Rightarrow$ (iii). If A is onto, then $\mathcal{R}(A) = W$ and is hence closed. Thus by the preceding theorem, $\mathcal{R}(A^*) = \mathcal{N}(A)^\perp$, which is also closed. Further (cf. Corollary 4.7.1), $\mathcal{N}(A^*) = \mathcal{R}(A)^\perp = \{\mathbf{0}\}$.

(iii) $\Rightarrow$ (i). If $\mathcal{N}(A^*) = \{\mathbf{0}\}$, and $\mathcal{R}(A^*)$ is closed, it follows from the preceding theorem that $\mathcal{R}(A) = \mathcal{N}(A^*)^\perp = W$. Thus A is onto.

(ii) $\Rightarrow$ (iii). By virtue of (4.7.11), it follows that $\mathcal{N}(A^*) = \{\mathbf{0}\}$. Let $\{v_n\}$ be a sequence in $D(A^*)$ such that $A^*v_n \to f$ in V^*. Then, again, by (4.7.11), it follows that

$$\|v_n - v_m\|_{W^*} \leq C\|A^*(v_n) - A^*(v_m)\|_{W^*}$$

and so $\{v_n\}$ is a Cauchy sequence. Let $v_n \to v$ in W^*. Since $G(A^*)$ is closed, it follows then that $v \in D(A^*)$ and that $A^*(v) = f$. Thus $\mathcal{R}(A^*)$

is closed.

(iii) $\Rightarrow$ (ii). Using the notation established in Proposition 4.7.4, $\mathcal{N}(A^*) = \{\mathbf{0}\}$ implies that $G^\perp \cap H^\perp = \{\mathbf{0}\}$ and $\mathcal{R}(A^*)$ closed implies that $G^\perp + H^\perp$ is closed. Hence, by Proposition 4.5.2, there exists a constant $C > 0$ such that for every $z \in G^\perp + H^\perp = \mathcal{R}(A^*) \times W^*$, there exist $a \in G^\perp$ and $b \in H^\perp$ such that $z = a + b$, and $\|a\|_{V^* \times W^*} \leq C\|z\|_{V^* \times W^*}$ and $\|b\|_{V^* \times W^*} \leq C\|z\|_{V^* \times W^*}$. Further, this decomposition is unique, since $G^\perp \cap H^\perp = \{\mathbf{0}\}$. Let $v \in D(A^*)$. Set $z = (A^*(v), \mathbf{0}) \in \mathcal{R}(A^*) \times W^* = G^\perp + H^\perp$. Then $a = (A^*(v), -v) \in G^\perp$ and $b = (\mathbf{0}, v) \in H^\perp$ and $a + b = z$. The inequality(4.7.11) now follows immediately. ■

Remark 4.7.3 A similar result can be stated and proved for the surjectivity of A^*. ■

Remark 4.7.4 If V and W are finite dimensional, then the ranges of A and A^* are automatically closed. In this case we have:

$$\begin{aligned} A \text{ onto} \quad &\Leftrightarrow \quad A^* \text{ one} - \text{one}, \\ A^* \text{ onto} \quad &\Leftrightarrow \quad A \text{ one} - \text{one}. \end{aligned}$$

However, in infinite dimensional Banach spaces, if we do not have information on the range being closed, we can only say:

$$\begin{aligned} A \text{ onto} \quad &\Rightarrow \quad A^* \text{ one} - \text{one}, \\ A^* \text{ onto} \quad &\Rightarrow \quad A \text{ one} - \text{one}. \end{aligned}$$

For instance, if $V = W = \ell_2$, and if

$$A(x) = \left(x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots\right),$$

then $A = A^*$ (check!) and A is clearly one-one, but not onto (cf. Example 2.3.3). ■

Remark 4.7.5 To show that A is onto, we usually use the relation (4.7.11). We assume that $A^*(v) = f$ and show that $\|v\|_{W^*} \leq C\|f\|_{V^*}$. This is called the method of *a priori* estimates. We do not worry about the existence of solutions to the equation $A^*(v) = f$, for a given f; if a solution exists, we obtain estimates for its norm. ■

Example 4.7.4 Let V and W be real Banach spaces and assume that W is reflexive. Let

$$a(.,.) : V \times W \to \mathbb{R}$$

be a bilinear form such that:
(i) $a(.,.)$ is continuous, *i.e.* there exists $M > 0$ such that, for all $v \in V$ and $w \in W$, we have

$$|a(v,w)| \leq M\|v\|_V\|w\|_W;$$

(ii) there exists $\alpha > 0$ such that, for all $w \in W$,

$$\sup_{v \in V, v \neq 0} \frac{|a(v,w)|}{\|v\|_V} \geq \alpha\|w\|_W;$$

(iii) there exists a constant $\beta > 0$ such that, for all $v \in V$, we have

$$\sup_{w \in W, w \neq 0} \frac{|a(v,w)|}{\|w\|_W} \geq \beta\|v\|_V.$$

Then, given $f \in V^*$ and $g \in W^*$, there exist unique elements $v_0 \in V$ and $w_0 \in W$ such that

$$\begin{array}{rcll} a(v_0,w) & = & < g,w >_{W^*,W} & \text{for all } w \in W; \\ a(v,w_0) & = & < f,v >_{V^*,V} & \text{for all } v \in V. \end{array}$$

To see this, define $A : V \to W^*$ by $< A(v), w >_{W^*,W} = a(v,w)$ for all $w \in W$. By the continuity of the bilinear form, A is a well defined and bounded linear operator. Since W is reflexive, we have $A^* : W \to V^*$, which is also a bounded linear operator (cf. Proposition 4.7.3) and it is easy to see that $< A^*(w), v >_{V^*,V} = a(v,w)$. By (ii), we have

$$\|A^*(w)\|_{V^*} \geq \alpha\|w\|_W.$$

Thus, by Theorem 4.7.2, we deduce that A is onto. By (iii), we have

$$\|A(v)\|_{W^*} \geq \beta\|v\|_V$$

whence A is one-one as well. In the same way, A^* is one-one and onto as well. Thus, there exist unique solutions to the equations

$$A(v_0) = g, \text{ and } A^*(w_0) = f$$

which proves the result. ■

4.8 Exercises

4.1 Show that a Banach space cannot have a basis whose elements form a countable set. Deduce that the space $\mathcal{P}$ of all polynomials in one variable cannot be complete for any norm.

4.2 (a) Let $\{a_n\}$ be a sequence of real numbers such that for any given real sequence $\{x_n\}$ such that $x_n \to 0$ as $n \to \infty$, the series

$$\sum_{n=1}^{\infty} a_n x_n$$

converges. Show that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. (Hint: Use Exercise 3.7(a)).
(b) Let $1 < p < \infty$. Let $\{a_n\}$ be a sequence of real numbers such that for all $x = (x_n) \in \ell_p$, the series $\sum_{n=1}^{\infty} a_n x_n$ is convergent. Show that $a = (a_n) \in \ell_{p^*}$.

4.3 (Numerical quadrature) Let $V = \mathcal{C}[0,1]$. For each positive integer n, define

$$\varphi_n(f) \;=\; \sum_{m=0}^{p_n} \omega_m^n f(x_m^n)$$

for all $f \in V$, where $\{x_m^n\}_{m=0}^{p_n}$ are given points in $[0,1]$ and $\{\omega_m^n\}_{m=0}^{p_n}$ are real numbers (called *weights*). Let

$$\varphi(f) \;=\; \int_0^1 f(t)dt$$

for $f \in V$.
(a) Show that $\varphi_n(f) \to \varphi(f)$ for every $f \in V$, as $n \to \infty$, if, and only if, the following conditions are verified:
(i) $\varphi_n(f_j) \to \varphi(f_j)$ as $n \to \infty$, for every integer $j \geq 0$, where $f_j(t) = t^j$;
(ii)

$$\sup_n \left\{ \sum_{m=0}^{p_n} |\omega_m^n| \right\} \;<\; \infty.$$

(cf. Exercise 2.13.)
(b) If $\omega_m^n \geq 0$ for all n and for all $0 \leq m \leq p_n$, show that the condition (ii) above is redundant.

(c) (Trapezoidal rule) Set $p_n = n$ and $x_m^n = m/n$ for $0 \leq m \leq n$. Let

$$\omega_m^n = \begin{cases} \frac{1}{n} & \text{if } m \neq 0, n \\ \frac{1}{2n} & \text{if } m = 0 \text{ or } n. \end{cases}$$

Show that $\varphi_n(f) \to \varphi(f)$ for all $f \in V$, as $n \to \infty$.

4.4 Let V be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a family of continuous linear operators on V. Assume that the following conditions hold:
(i) $S(0) = I$, the identity operator on V.
(ii) For all $t_1 \geq 0$ and $t_2 \geq 0$, we have

$$S(t_1 + t_2) = S(t_1) \circ S(t_2).$$

(iii) For all $x \in V$,

$$\lim_{t \downarrow 0} S(t)(x) = x.$$

Then, we say that $\{S(t)\}_{t\geq 0}$ is a c_0-*semigroup of operators* on V.
(a) Let $A \in \mathcal{L}(V)$. Define $S(t) = \exp(tA)$ (cf. Exercise 2.30) for $t \geq 0$. Show that $\{S(t)\}_{t\geq 0}$ forms a c_0-semigroup of operators on V.
(b) Let V denote the space of all bounded and uniformly continuous real valued functions on $\mathbb{R}$ provided with the usual 'sup-norm'. For $t \geq 0$, define $S(t)$ by

$$S(t)(f)(\tau) = f(t + \tau)$$

for $\tau \in \mathbb{R}$. Show that $S(t) \in \mathcal{L}(V)$ for each $t \geq 0$ and that $\{S(t)\}_{t\geq 0}$ is a c_0-semigroup of operators on V.

4.5 Let V be a Banach space and let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on V.
(a) Show that there exists $M > 0$ (which, without loss of generality, can be chosen to be greater than, or equal to, unity) and $\eta > 0$ such that, for all $0 \leq t \leq \eta$, we have

$$\|S(t)\| \leq M.$$

(b) Deduce that if $\omega = \eta^{-1} \log M \geq 0$, then

$$\|S(t)\| \leq Me^{\omega t}$$

for all $t \geq 0$.
(c) The semigroup is said to be *exponentially stable* if we can find $M > 0$

and $\omega < 0$ such that the preceding inequality is true. Show that a c_0-semigroup $\{S(t)\}_{t\geq 0}$ is exponentially stable if, and only if, there exists a $t_0 > 0$ such that $\|S(t_0)\| < 1$.
(d) Prove that for every $x \in V$, fixed, the mapping $t \mapsto S(t)x$ is continuous from the interval $[0, \infty)$ into V.
(e) Prove that

$$\lim_{h\downarrow 0} \frac{1}{h} \int_t^{t+h} S(\tau)(x)\, d\tau \;=\; S(t)(x)$$

for all $t \geq 0$ and for every $x \in V$.

4.6 Let V be a real Banach space and let $a(.,.) : V \times V \to \mathbb{R}$ be a continuous bilinear form (cf. Example 4.7.4). Assume that for every $x \in V$, $x \neq \mathbf{0}$,

$$a(x, x) \;>\; 0.$$

Let $T : V \to V$ be a linear map such that, for all $x \in V$ and $y \in V$, we have

$$a(T(x), y) \;=\; a(x, T(y)).$$

Show that $T \in \mathcal{L}(V)$.

4.7 Let V and W be Banach spaces and let $\{f_i\}_{i\in I}$ (where I is an indexing set) be a collection of continuous linear functionals on W which separates points in W. Let $T : V \to W$ be a linear map. If $f_i \circ T$ is continuous for each $i \in I$, show that $T \in \mathcal{L}(V, W)$.

4.8 Let X, Y and Z be Banach spaces. Let $T \in \mathcal{L}(X, Z)$ and $A \in \mathcal{L}(Y, Z)$. Assume that for every $x \in X$, there exists a unique $y \in Y$ such that $A(y) = T(x)$. Define $B : X \to Y$ by $Bx = y$. Show that $B \in \mathcal{L}(X, Y)$.

4.9 Let V and W be Banach spaces. let $T \in \mathcal{L}(V, W)$. We say that $S \in \mathcal{L}(W, V)$ is a *left inverse* of T if $S \circ T = I_V$, where I_V is the identity operator on V. Show that T has a left inverse if, and only if, T is injective and $\mathcal{R}(A)$ is closed and complemented in W.

4.10 (a) Let W be a Banach space and let $T : D(T) \subset W \to W$ be a closed and densely defined linear operator. Set $V = D(T)$ and define, for $x \in V$,

$$\|x\| \;=\; \|x\|_W + \|T(x)\|_W.$$

Show that V is a Banach space for this norm.
(b) If V is also a Banach space for some other norm $\|.\|_V$, and if this norm is such that both the inclusion map of V into W and the map T are in $\mathcal{L}(V, W)$, show that there exists a constant $C > 0$ such that, for all $x \in V$,

$$\|x\|_V \leq C\left(\|x\|_W + \|T(x)\|_W\right).$$

4.11 Let V and W be Banach spaces and let $T \in \mathcal{L}(V, W)$ be surjective. Show that T is injective if, and only if, there exists a constant $c > 0$ such that, for all $v \in V$, we have

$$\|T(v)\|_W \geq c\|v\|_V.$$

4.12 Let c_F be the set of all real sequences such that all but a finite number of terms are zero provided with the norm $\|.\|_\infty$. Define $T : c_F \to c_F$ by

$$T(x) = \left(x_1, \frac{x_2}{2}, \cdots, \frac{x_n}{n}, \cdots\right)$$

where $x = (x_n) \in c_F$. Show that T is a bijection and that $T \in \mathcal{L}(c_F)$. Show, however, that T is not an isomorphism. Why does this not contradict Corollary 4.4.1?

4.13 Let V be a Banach space and let $\{S(t)\}_{t \geq 0}$ be a c_0-semigroup of operators on V. Define

$$D(A) = \left\{x \in V \mid \lim_{h \downarrow 0} \frac{1}{h}(S(h)(x) - x) \text{ exists}\right\}$$

and, for $x \in D(A)$,

$$A(x) = \lim_{h \downarrow 0} \frac{1}{h}(S(h)(x) - x).$$

The operator $A : D(A) \subset V \to V$ is called the *infinitesimal generator* of the semigroup $\{S(t)\}_{t \geq 0}$. If $A \in \mathcal{L}(V)$, show that it is the infinitesimal generator of the semigroup $\{\exp(tA)\}_{t \geq 0}$.

4.14 Let V and $\{S(t)\}_{t \geq 0}$ be as in Exercise 4.4(b). Show that the infinitesimal generator of the semigroup is given by:

$$D(A) = \{f \in V \mid f' \text{ exists and } f' \in V\},$$

and

$$A(f) = f'$$

where f' denotes the derivative of f.

4.15 Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on a Banach space V. Let $A : D(A) \subset V \to V$ be its infinitesimal generator. For any $x \in V$, show that

$$\int_0^t S(\tau)(x)\, d\tau \in D(A)$$

and that

$$A\left(\int_0^t S(\tau)(x)\, d\tau\right) = S(t)(x) - x.$$

4.16 Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on a Banach space V. Let $A : D(A) \subset V \to V$ be its infinitesimal generator.
(a) Let $x \in D(A)$. Show that $S(t)(x) \in D(A)$ for all $t > 0$ and that

$$\frac{d}{dt}(S(t)(x)) = A(S(t)(x)) = S(t)(A(x)).$$

(b) If $x \in D(A)$ and if $0 \leq t_2 < t_1$, show that

$$S(t_1)(x) - S(t_2)(x) = \int_{t_2}^{t_1} S(\tau)(A(x))\, d\tau = \int_{t_2}^{t_1} A(S(\tau)(x))\, d\tau.$$

4.17 Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on a Banach space V. Let $A : D(A) \subset V \to V$ be its infinitesimal generator. Show that A is densely defined and closed.

4.18 Let V be a Banach space and let $A : D(A) \subset V \to V$ be a linear operator. We say that a map $t \mapsto u(t)$ from $[0, \infty)$ into V is a solution to the initial value problem:

$$\begin{aligned} \tfrac{du(t)}{dt} &= A(u(t)),\ t > 0, \\ u(0) &= x \end{aligned}$$

if $u(t) \in D(A)$ for all $t > 0$ and if it verifies the above equations.
(a) If $x \in D(A)$, and if A is the infinitesimal generator of a c_0-semigroup of operators $\{S(t)\}_{t\geq 0}$ on V, show that the only solution to the above initial value problem is given by $u(t) = S(t)(x)$ for $t \geq 0$. (Hint: Clearly $u(t)$ defined thus is a solution by the Exercise 4.16; to show uniqueness, differentiate the map $\tau \mapsto S(t-\tau)(u(\tau))$, $\tau \in (0, \infty)$.)
(b) If V is a Banach space and if $\{S_1(t)\}_{t\geq 0}$ and $\{S_2(t)\}_{t\geq 0}$ are two c_0-semigroups of operators on V which have the same infinitesimal generator, show that $S_1(t) = S_2(t)$ for all $t \geq 0$.

(c) Deduce that the only semigroups on V whose infinitesimal generators are in $\mathcal{L}(V)$ are of the form $\{\exp(tA)\}_{t\geq 0}$ with $A \in \mathcal{L}(V)$.

4.19 Let $\{S(t)\}_{t\geq 0}$ be a c_0-semigroup of operators on a Banach space V. Let $A : D(A) \subset V \to V$ be its infinitesimal generator. Assume further that, for all $t \geq 0$, $\|S(t)\| \leq 1$. (Such a semigroup is called a *semigroup of contractions*).
(a) Let $\lambda > 0$. For $x \in V$, define

$$R(\lambda)(x) = \int_0^\infty e^{-\lambda t} S(t)(x)\, dt$$

(cf. Exercise 3.15). If $x \in D(A)$, show that $R(\lambda)(x) \in D(A)$ for all $\lambda > 0$ and that

$$R(\lambda)(A(x)) = A(R(\lambda)(x)).$$

(b) Show that for all $\lambda > 0$,

$$(\lambda I - A)(R(\lambda)(x)) = x$$

for all $x \in V$ and that

$$R(\lambda)((\lambda I - A)(x)) = x$$

for all $x \in D(A)$, where I is the identity map on V.

Remark 4.8.1 Exercise 4.19 shows that if A is the infinitesimal generator of a semigroup of contractions, then the (unbounded) linear opeartor $\lambda I - A$ is invertible and that its inverse is $R(\lambda)$ which is a bounded linear operator defined on V. Thus, together with Exercise 4.17, we see that in order that an (unbounded) linear operator A be the infinitesimal generator of a semigroup of contractions, it has to be densely defined, closed and for all $\lambda > 0$, $\|(\lambda I - A)^{-1}\| \leq 1/\lambda$. In fact these conditions are also sufficient for A to be the infinitesimal generator of a semigroup of contractions. This is the content of the famous *Hille-Yosida theorem*. Generalizations to other c_0-semigroups also exist. The usefulness of this result stems from the fact that many partial differential equations of the evolution type (for instance, the heat, wave and Schrödinger equations) can be cast in the form of an initial value problem as stated in Exercise 4.18 involving an unbounded linear operator and so the existence of uniqueness of solutions will follow from the the fact that the operator is the infinitesimal generator of a c_0-semigroup. For more details see, for instance Kesavan [3]. ■

4.20 Let V and W be Banach spaces and let $A : D(A) \subset V \to W$ be a closed operator. Let $B \in \mathcal{L}(V, W)$. Show that $(A+B) : D(A) \subset V \to W$ is also closed.